

TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 4

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Homework 4.1 (Absolute convergence of infinite products). Let $(a_j)_{j \in \mathbf{N}}$ be a sequence in $\mathbf{C}$. We say the infinite product $\prod_{j=1}^{\infty} a_j$ converges absolutely if there exists $j_0 \in \mathbf{N}$ such that $a_j \neq 0$ for every $j \geq j_0$ and $(|\log a_j|)_{j \geq j_0}$ is summable, i.e.

$$\sum_{j \geq j_0} |\log a_j| < \infty.$$

- a. Show absolute convergence for infinite products implies their convergence.
- b. Show a product of the form $\prod_{j=1}^{\infty} (1 + b_j)$ converges absolutely if and only if

$$\sum_{j=1}^{\infty} |b_j| < \infty.$$

Solution. a. This is an immediate consequence of Lemma 3.3 since the convergence of $\sum_{j \geq j_0} |\log(a_j)|$ implies $|\log a_j| \rightarrow 0$ as $j \rightarrow \infty$ and thus $a_j \rightarrow 1$ as $j \rightarrow \infty$.

- b. Since $\log(1) = 0$ and $\log'(1) = 1$ it follows that for $|z|$ small enough,

$$\frac{1}{2}|z| \leq |\log(1 + z)| \leq \frac{3}{2}|z|.$$

Since the convergence of both $\sum_{j=1}^{\infty} |b_j|$ and $\prod_{j=1}^{\infty} (1 + b_j)$ imply $b_j \rightarrow 0$ as $j \rightarrow \infty$, we conclude in both cases there exists $j_0 \in \mathbf{N}$ such that for every $j \geq j_0$,

$$\frac{1}{2}|b_j| \leq |\log(1 + b_j)| \leq \frac{3}{2}|b_j|.$$

The claim follows from the definition of absolute convergence of infinite products.

Homework 4.2 (Examples of infinite products*). Examine if the following infinite products exist in the sense of Definition 3.1. If so, calculate their value¹.

- a. $\prod_{n=1}^{\infty} \left[1 - \frac{1}{(n+1)^2} \right].$
- b. $\prod_{n=1}^{\infty} \left[1 - \frac{1}{n} \right]$
- c. $\prod_{n=3}^{\infty} \frac{n^2 - 4}{n^2 - 1}.$
- d. $\prod_{n=1}^{\infty} \frac{(1 + n^{-1})^2}{1 + 2n^{-1}}$

Homework 4.3 (Diverging products). Let $(a_j)_{j \in \mathbf{N}}$ form a sequence in $[0, +\infty)$ with the property $\sum_{j=1}^{\infty} (1 - a_j) = \infty$. Show² that

$$\lim_{n \rightarrow \infty} \prod_{j=1}^n a_j = 0.$$

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¹**Hint.** In all examples, you can directly calculate the value of the partial products.

²**Hint.** Use the inequality $t \leq e^{t-1}$ for every $t \in \mathbf{R}$.

Solution. From the hint we deduce

$$0 \leq \prod_{j=1}^n a_j \leq \prod_{j=1}^n e^{a_j-1} = \exp\left[-\sum_{j=1}^n (1-a_j)\right].$$

Using the assumption the claim follows by letting $n \rightarrow \infty$.

Homework 4.4 (A useful criterion for the convergence of infinite products). Let $(a_j)_{j \in \mathbb{N}}$ be a sequence in $\mathbb{C}$. Assume $\sum_{j=1}^{\infty} |a_j|^2 < \infty$. Show $\prod_{j=1}^{\infty} (1+a_j)$ converges if and only if $\sum_{j=1}^{\infty} a_j$ converges. Conclude the product $\prod_{j=1}^{\infty} (1+z/j)$ converges if and only if $z=0$.

Solution. The hypothesis on square-summability of $(a_j)_{j \in \mathbb{N}}$ implies there exists $j_0 \in \mathbb{N}$ such that $|a_j| \leq 1/2$ for every $j \geq j_0$. Let us set

$$\theta_j := \frac{\log(1+a_j) - a_j}{a_j^2} = \sum_{n=2}^{\infty} (-1)^{n+1} \frac{a_j^{n-2}}{n}.$$

Since $a_j \rightarrow 0$ as $j \rightarrow \infty$ it follows $\theta_j \rightarrow -1/2$ as $j \rightarrow \infty$. In particular, the sequence $(\theta_j)_{j \in \mathbb{N}}$ is bounded. From this, it easily follows the series

$$\sum_{j=j_0}^{\infty} \theta_j a_j^2 = \sum_{j=j_0}^{\infty} (\log(1+a_j) - a_j) \quad (4.1)$$

converges. Thus on the one hand the convergence of $\sum_{j=1}^{\infty} a_j$ implies the convergence of $\sum_{j=j_0}^{\infty} \log(1+a_j)$. Taking the exponential in the latter identity, we infer that in this case also $\prod_{j=j_0}^{\infty} (1+a_j)$ converges to a non-zero limit. In particular the infinite series $\prod_{j=1}^{\infty} (1+a_j)$ converges. On the other hand, the convergence of the latter product implies for a suitably large $j_1 \geq j_0$ that

$$\log\left[\prod_{j=j_1}^{\infty} (1+a_j)\right] = \sum_{j=j_1}^{\infty} \log(1+a_j),$$

cf. the proof of Lemma 3.3. Hence due to (4.1) the convergence of the product implies also the convergence of $\sum_{j=j_1}^{\infty} a_j$, which yields the claim.

Applying this result to the product $\prod_{j=1}^{\infty} (1+z/j)$ — for which the above assumptions hold for every $z \in \mathbb{C}$ — we see that it converges if and only if $z \sum_{j=1}^{\infty} 1/j$ converges. This is the case if and only if $z=0$.